CS 331, Fall 2026 Today: Asymptotics

Mini-lecture 2 (8/26)

Asymptotics

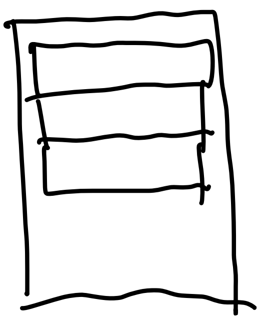
Goal: compare $f(n)$, $g(n)$ (resources used when input size $= n$)

as $\to \infty$

Scope: $n \in \mathbb{N}$, $f(n), g(n) > 0$

input size resources used

Belief: "constant" may come from many sources



server quality

def f:

language overhead

. . .

Asymptotic improvement can only come from algo change

Formal def'n: "Big O"

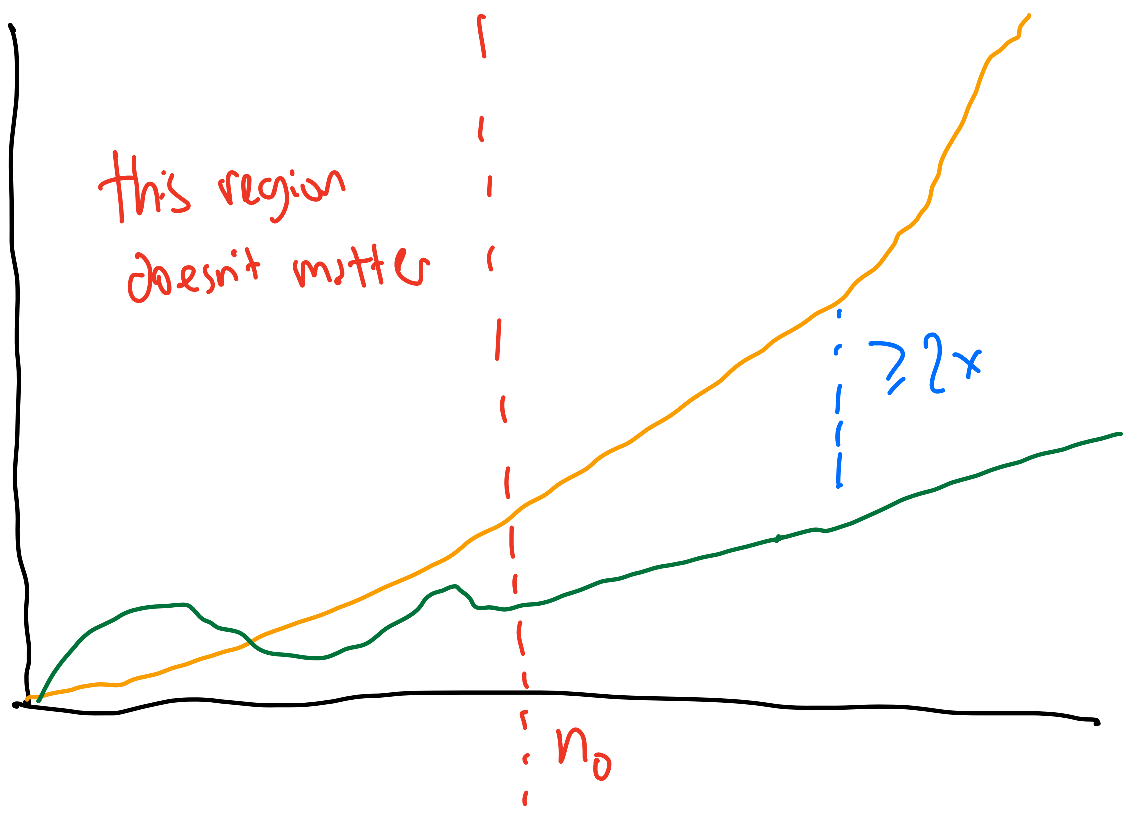
Let f and g be functions of $n \in \mathbb{N}$

We say $f(n) = O(g(n))$ if:

$\exists\ C > 0,\ n_0 \in \mathbb{N}$ s.t.

$$0 < f(n) \leq C\, g(n) \quad \forall\, n \geq n_0$$

a.k.a. (when it exists) $\displaystyle\lim_{n \to \infty} \frac{f(n)}{g(n)} \leq C$



Why n_0?

Pretty much just because $\log(1) = 0$...

Cannot say $n = O(n\log(n))$ otherwise!

Claim: If $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ (positive)

Then may as well take $n_0 = 1$

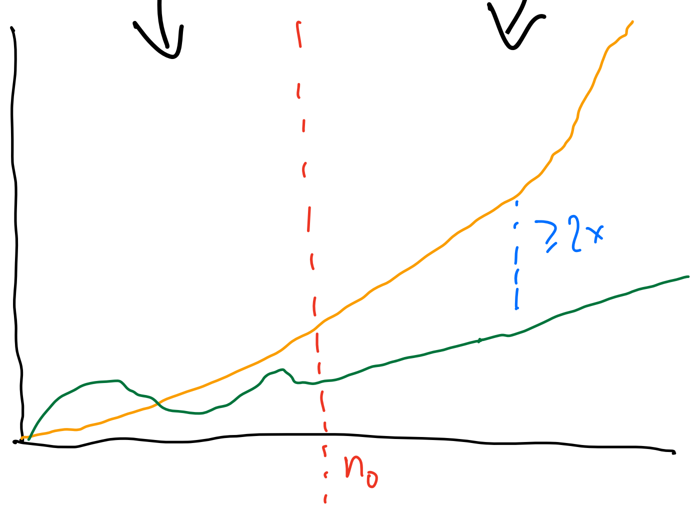
Proof: Suppose $\exists\, n_0, C$ for original def.

Then $\forall n \geq n_0' = 1$, $f \leq C' g$

$$C' = \max\left(\max_{n \leq n_0} \frac{f(n)}{g(n)},\ C \right)$$

Biggest ratio $\leq n_0$ is just some constant!

(unaffected as $n \to \infty$)



$\boxed{\text{Example}}$ Suppose f, g are polynomials

$$f(n) = A n^c + \ldots + \text{constant} \qquad A > 0$$

$$g(n) = B n^d + \ldots + \text{constant} \qquad B > 0$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & c < d \\ \frac{A}{B} & c = d \\ \infty & c > d \end{cases}$$

Quiz: Which of above cases has $f = O(g)$?

Quiz: O is an "ordering" (satisfies transitive property)

$$\underbrace{f = O(g)}_{n_0,\, C},\ \underbrace{g = O(h)}_{n_0',\, C'} \Rightarrow \underbrace{f = O(h)}_{???}$$

The Zoo

$f = \Omega(g)$ a.k.a. $g = O(f)$

$f = \Theta(g)$ a.k.a. $f = O(g)$ & $g = O(f)$

$f = o(g)$ a.k.a. $f \neq \Omega(g)$

$f = \omega(g)$ a.k.a. $f \neq O(g)$

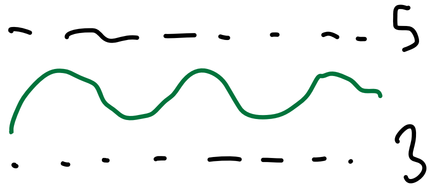
Quiz: Limit def'ns? What is $\lim_{n \to \infty} \frac{f}{g}$

Constant: $f = O(g), \Omega(g), \Theta(g)$

0: $f = o(g), O(g)$

∞: $f = \Omega(g), \omega(g)$

Warning: limit doesn't always exist... but it will in this class.



$= O(1)$

What will we encounter? "natural f"

- monotone increasing
- w/ no exceptions,

polylogarithmic $\ll$ polynomial $\ll$ exponential

Ex. Which is bigger asymptotically?

$$f(n) = 2^{\sqrt{\log(n)}} \quad \text{or} \quad g(n) = n^{0.1}$$

Tip: applying 1-to-1 increasing transform $\underbrace{t}_{\log,\ x^2 \text{ for } x>0,\ \exp}$ cannot change asymptotic relationship!

$$f(n) = O(g(n)) \iff f(t(n)) = O(g(t(n)))$$

Ex. Suppose you believe $n^a = o(b^n)$ $\quad b > 1, \ a > 0$

Why is $\log^c(n) = o(n^a)$? $\quad c > 0$

$t = \log(n)$

Theorem: For all $f, g : \mathbb{N} \to \mathbb{R}$

$$f(n) + g(n) = \Theta(\max(f(n), g(n)))$$

Ex. $n^2 + n = \Theta(n^2)$

Algo:

$$\left.\begin{array}{ll} \text{Step 1} & O(n^2) \\ \text{Step 2} & O(n^3) \\ \text{Step 3} & O(\log(n)) \end{array}\right\} \quad O(n^3)$$

Warning: do not use non-constant # times!

Proof:

$$\max(f, g) \leq f + g \leq 2 \cdot \max(f, g)$$

$\blacksquare$

Challenge: $\log(n!) = \Theta(f(n))$ for simple function f. What is it?

Ideas: $n! = 1 \cdot 2 \cdot \ldots \cdot n$ will beat eventually

$2^n = 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2$

Proof: We claim $f(n) = n\log(n)$

UB: $n! \leq n \cdot n \cdot n \cdot \ldots \cdot n = n^n = e^{n\log(n)}$

LB: $n! \geq \frac{n}{2} \cdot (\frac{n}{2}+1) \cdot \ldots \cdot n \geq (\frac{n}{2})^{\frac{n}{2}}$

$\log(n!) \geq \frac{n}{2}\log(\frac{n}{2}) = \frac{n}{2}\log(n) - O(n)$

"Proof": $\log(n!)$

$= \log(1) + \log(2) + \ldots + \log(n)$

$\approx \int_1^n \log(x)\,dx = x\log x - x \Big|_1^n = O(n\log(n))$